1. Algebra of complex numbers. Integration and differentiation of functions of complex variables.

Problem 1.1

Provide a geometric description of the following sets in the complex plane and derive it geometrically and algebraically. Reduce the equations for the boundaries to the canonical form.

1. $2 < |z-i| < 4$.
2. $|z-4i| + |z+4i| = 10$.
3. $\mathrm{Im}\, \frac{1}{z} = 1$.

Problem 1.2

Let ε be arbitrary n-th rooth of unity (not equal to 1). Prove the following equality

$$1 + 2\varepsilon + 3\varepsilon^2 + \ldots + n\varepsilon^{n-1} = \frac{n}{\varepsilon - 1}.$$

Problem 1.3

Determine the images

1. of a line $\mathrm{Im}\, z = 1$ under the map $z \to w(z) = z^3 + 3z - i$.
2. of a circle $|z-i| = 1$ under the map $z \to w(z) = \frac{1}{z-2i}$.

Problem 1.4

Do the following functions of $z = x + iy$ satisfy Cauchy–Riemann conditions?

1. $w(z) = x^2 + y^2$.
2. $w(z) = x^2 - y^2 + 2ixy$.
3. $w(z) = \frac{1}{x+iy}$.

Problem 1.5

Recover an analytic function $f(z = x + iy)$ satisfying the following equations

1. $|f| = e^{r^3 \cos 3\varphi}$ with $z = re^{i\varphi}$.
2. $\arg f = xy$.

Problem 1.6

Find all harmonic functions of the following form

1. $u = \varphi(x^2 - y^2)$.
2. $u = \varphi\left(\frac{y}{x}\right)$.

Problem 1.7

Calculate the integral along the unit circle $\mathcal{C}$, centered at $z = 0$

1. $\int_{\mathcal{C}} z dz$.
2. $\int_{\mathcal{C}} z^* dz$.

Problem 1.8

Calculate the integral

$$\int_{\mathcal{C}} \frac{ydx - xdy}{x^2 + y^2}$$

1. along the unit circle $\mathcal{C}$, centered at $z = 0$, counter–clockwise.
2. along the unit circle $\mathcal{C}$, centered at $z = 2$, counter–clockwise.

Problem 1.9

Consider a function of a natural number n defined by the following integral

$$p(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} dz z^{-1-n} \prod_{k=1}^{\infty} \frac{1}{1 - z^k},$$

where $\mathcal{C}$ is a circle of a radius smaller than 1, centered at $z = 0$ and oriented counter–clockwise.

1. Show that $p(n)$ is a natural number.
2. Evaluate $p(1)$ and $p(4)$.

Problem 1.10

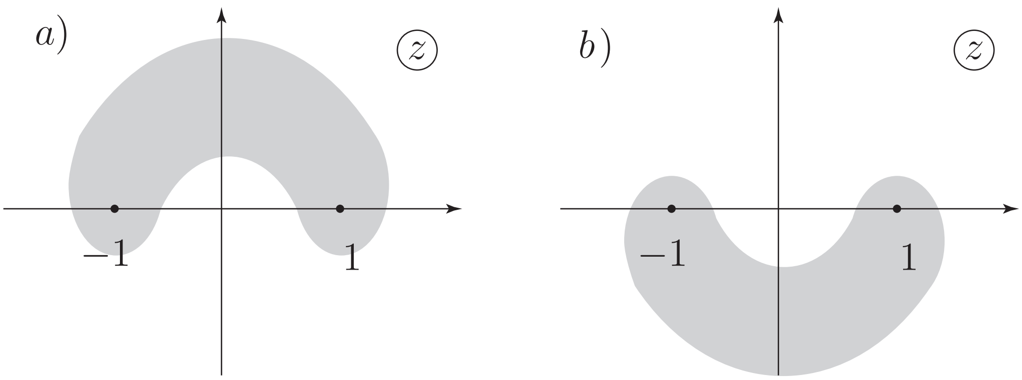

FIG. 1: Region $\mathcal{D}$ for the Problem 1.10.

Consider the function $y(z)$ satisfying $y(1) = 0$ and $y'(z) = \frac{1}{2z}$ in the region $\mathcal{D}$. Evaluate $y(-1)$ for

1. the region $\mathcal{D}$ shown on the Fig. 1a.
2. the region $\mathcal{D}$ shown on the Fig. 1b.